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On structure of cogroups

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Abstract

Some results on the characterization of D-hypergroups, Utumi cogroups and cogroups are given, and a general viewpoint is tackled.

1. D-hypergroups and cogroups

A D-hypergroups [4] is a classical natural hypergroup:

Let G be a (multiplicative) group and g a subgroup of G, the set G/g of the right cosets is a hypergroup by the natural multivalued operation (hyperoperation) as follows:

$$agbg = \{cg\}_{c \in agbg} = \{cg\}_{c \in agb}.$$

A hypergroup which is isomorphic to this kind of hypergroup is a D-hypergroup.

Question. Is it possible to have D-hypergroup's characterizations in a hypergroup theory?

In 1940 Eaton [1] introduced cogroups for this question. But in 1949 Utumi [9] gave an example of a cogroup which was not a D-hypergroup.

In 1988 Haddad and Sureau [2] showed that the class of all D-hypergroups was not an elementary class (in the sense of model theory) and that the first-order theory of D-hypergroups was not finitely axiomatizable.

But there exists, all the same, one characterization of a D-hypergroup and according to Krasner [5, 6], a criterion for a cogroup to be a D-hypergroup and a method to construct a new cogroup starting from an initial cogroup are given by Utumi [9].

Lastly, in 1988, a criterion was established, by Haddad and Sureau [3], for a cogroup to be obtainable by Utumi's method starting from a D-hypergroup.

Now a general viewpoint is undertaken.

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2. Two definitions

From elementary proprieties of D-hypergroups it is natural to introduce the following two notions:

Cogroup (Eaton [1]). A (right) cogroup is a (multiplicative) hypergroup, H, with the following three conditions:

(1) $(\exists e \in H)(\forall X \in H)[Xe = X]$, *e is the right scalar unit.*

(2) $\forall (X, Y, Z) \in H^3$, $XY \cap XZ \neq \emptyset \Rightarrow eY = eZ$.

(3) $\forall (X, Y, Z) \in H^3$, $|XZ| = |YZ|$ ($|XZ|$ is the cardinality of XZ).

Note. We see that $\{aX\}_{X \in H}$ is a partition of H for all a of H.

Multiplicator (Utumi [9]). A permutation, σ, of a hypergroup H is a multiplicator (of H) iff: $\forall (X, Y) \in H^2$, $\sigma(XY) = \sigma(X)Y$.

We denote by M the group of all the multiplicators of H.

3. A general viewpoint

We denote by H a cogroup (unit: e) and by Γ the group of all the permutations of H. We consider

$$K = \{\sigma \in \Gamma;\ (\forall X \in H)\ [\sigma(eX) = \sigma(e)X]\},$$

$$k = \{\sigma \in \Gamma;\ (\forall X \in H)\ \sigma(eX) = eX\}$$

$$= \{\sigma \in K;\ \sigma(e) = e\}.$$

We have the following proprieties and results:

— it is obvious that k is a subgroup of Γ and $M \subset K$.

— *if* $\alpha \in K$ *then* $M\alpha k \subset K$: For some $m \in k$, $\beta \in k$ and $X \in H$ we have, $m\alpha\beta(eX) = m\alpha(eX) = m(\alpha(e)X) = (m\alpha(e))X = (m\alpha\beta(e))X$,

— $K(e) = H$: Because in a cogroup, for some $a \in H$, there exists a permutation $\sigma \in \Gamma$ such that $a = \sigma(e)$ and the partition $\{eX\}_{X \in H}$ has for image $\{[\sigma(e)]X\}_{X \in H}$ (we recall we have $|eX| = |aX|$) σ is in K and we have $K(e) = H$.

— *K is a group iff* $K = M$: If $K = M$ then K is group. Conversely for some $a \in H$ there exists $\sigma \in K$ such that $\sigma(e) = a$, if K is a group for some $X \in H$ and $\alpha \in K$, we have $\alpha\sigma \in K$ and

$$\alpha(aX) = \alpha(\sigma(e)X) = \alpha(\sigma(eX)) = \alpha\sigma(eX)$$

$$= (\alpha\sigma(e))X = (\alpha(a))X$$

then $\alpha \in M$ and $K = M$.

— $(\forall X \in H), k(X) = eX$: Because in k there are all transpositions $\tau_{XX'}$ which change X with some X' of eX we have $eX \subset k(X) \subset k(eX) = eX$.

— $(\forall \alpha \in k)(\forall \beta \in K)$, $\alpha(e) = \beta(e) \Leftrightarrow \alpha k = \beta k$: If $\alpha k = \beta k$ then $\alpha(e) = \alpha k(e) = \beta k(e) = \beta(e)$. Conversely for some $X \in H$ if $\alpha(eX) = \alpha(e)X = \beta(e)X = \beta(eX)$ then $\beta^{-1}\alpha(eX) = eX$ and $\beta^{-1}\alpha \in k$ and $\beta k = \alpha k$, follows.

— Therefore, we can build the following commutative diagram:

$$\begin{tikzcd}
K \arrow[r] & K(e) = H \\
\alpha \arrow[r, squiggly] \arrow[d, "s"'] & \alpha(e) \\
K/k \arrow[ur, "\phi"'] &
\end{tikzcd}$$

where s is the canonical surjection and ϕ is bijective, and we can define, by ϕ one structure of cogroup on $K/_k$: $\alpha k.\beta k = \phi^{-1}(\alpha(e)\beta(e))$.

We discern three cases:

(1) $K = M$ then $k = M_e = \{\alpha \in M; \sigma(e) = e\}$ and, by the Krasner–Utumi criterion, H is a D-hypergroup ($\simeq M/_{M_e}$).

(2) $M \neq K = Mk$: We have $H = Mk(e) = M(e)$ and, by the Haddad–Sureau criterion, H is isomorphic to a cogroup obtainable by Utumi.s method starting from a D-hypergroup $M/_{M_e}))((Mk/_{k},\cdot) \simeq (M/_{M_e},*))$ and H is not a D-hypergroup.

(3) $M \neq K \neq Mk$, open problem!

During September 1992, at Messine University an answer was given and will be published at a later date.

In fact we consider the subgroup L of Γ generated by K and the stability subgroup $l = L_e = \{\sigma \in L; \sigma(e) = e\}$ of e in L, and we define a hypermultiplication '.' on $L/_l$ such that $(L/_{l},*)$ is an isomorphic cogroup of H.

Conversely if L is a group and k, l $(k \subset l)$ are two subgroups of L we give necessarily and sufficient conditions on k, l, L and a subset K of k-closets of L such that $L/_l$ has a cogroup's structure.

Appendix

Krasner–Utumi's criterion (Krasner [5,6] and Utumi [9]). Let H be a cogroup (with the right scalar unit: e). Then H is a D-hypergroup iff a group G of multiplicators of H exist such that $G(e) = H$ and the stability sub-group, g, of $e(g = \{\sigma \in G; \sigma(e) = e\})$ satisfies $g(x) = ex$ for all x of H (then $H \simeq G/g$).

Utumi's method [9]. Let H be a cogroup and R an equivalence relation on H. We give H the following hyperoperation:

$$X_* Y = X c\ell_R(Y).$$

Then $(H, *)$ is a cogroup iff R satisfies the three conditions:

(1) $c\ell(e) = e$,

(2) $e c\ell(X) = c\ell(X)$,

(3) $c\ell(X c\ell(Y)) = c\ell(X) c\ell(Y)$.

Haddad–Sureau's criterion [3]. Let H be a cogroup (unit: e). H is isomorphic to a cogroup obtainable by Utumi's method starting from a D-hypergroup iff a group G of multiplicators of H exist such that $G(e) = H$ (G is transitive on H) (then $H \simeq (G/g, *)$, where $g = \{\sigma \in G;\ \sigma(e) = e\}$).

Note. Cogroups which are not obtained by Utumi's method starting from a D-hypergroup are exhibited by Haddad and Sureau [3].

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